10

Non-linear Relationships

There are many families of curves that have ***non-linear relationships***.
Non-linear relationships are relationships that when graphed produce graphs that are not straight lines.
In this chapter the following curves will be studied:
parabolas, rectangular hyperbolas, circles and exponential functions.

Parabolas

EXERCISE 10A

Parabolas are the curves formed by graphing quadratic equations.
Quadratic equations are equations that have 2 as the highest power.

Examples
The following are quadratic equations.

$y = x^2$	$y = 3x^2 - 4$
$h = 7t^2 + 5t - 1$	$a = 5 - 6b^2$

The following are ***not*** quadratic equations.

$y = x + 1$	$y = 8x^3 - 4x^2 + 3$
$a = 5 - 6b^4$	$h = 7t^2 + 5t^3 - t$

1. State which of the following are quadratic equations.

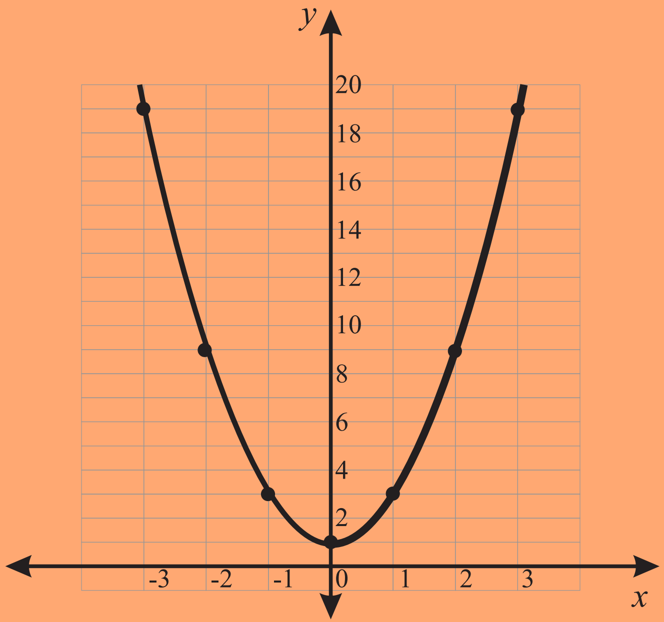
A $y = 5x$	**B** $y = 2x^2$
C $a = 6 + b^2$	**D** $m = 3n^3 - n^2$
E $y = 6x + 2$	**F** $v = 3t^2 - 5$
G $h = 2c^4 + 3c^2 - 6c$	**H** $y = 7 - 4x + 5x^2$
I $B = 2A^3 + 4A^2 - 5A$	**J** $d = 4r^2 + 2r - 7$
K $y = 2x^2 + 5x + 8$	**L** $z = 7 + 2y^5$

When a quadratic equation is graphed, the resultant curve is called a ***parabola***.

Example $y = 2x^2 + 1$

This equation can be graphed by completing a tables of values, plotting the points and connecting them with a smooth curve.

x	-3	-2	-1	0	1	2	3
y	19	9	3	1	3	9	19



2. For each of the following quadratic equations, complete the table of values shown, plot the points on a graph and connect the points with a smooth curve.

(a) $y = x^2 - 2x$

x	-2	-1	0	1	2	3	4
y							

(b) $y = x^2 + 4x + 1$

x	-5	-4	-3	-2	-1	0	1
y							

3. (a) Using a graphics calculator, or otherwise, plot on the same axes the following sets of graphs.

A $y = x^2$ **B** $y = 2x^2$ **C** $y = 3x^2$

(b) Based on these three graphs, what effect does the coefficient of x^2, a, have on the graph $y = ax^2$, if $a > 1$?

4. (a) Using a graphics calculator, or otherwise, plot on the same axes the following sets of graphs.

A $y = x^2$ **B** $y = \frac{1}{2}x^2$ **C** $y = \frac{1}{3}x^2$

(b) Based on these three graphs, what effect does the coefficient of x^2, a, have on the graph $y = ax^2$, if $0 < a < 1$?

5. (a) Using a graphics calculator, or otherwise, plot on the same axes the following sets of graphs.

A $y = -x^2$ **B** $y = -\frac{1}{2}x^2$ **C** $y = -3x^2$

(b) Based on these three graphs, what effect does the coefficient of x^2, a, have on the graph $y = ax^2$, if $a < 0$?

Summary of the shape of parabolas.
It can be seen that all parabolas have the same basic shape.

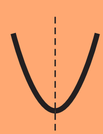
If the coefficient of x^2 is positive, the curve is in the shape of a 'cup'.

If the coefficient of x^2 is negative, the curve is in the shape of a 'hat'.

If the coefficient of x^2 is greater than 1, the curve will be thinner.

If the coefficient of x^2 is a fraction less than 1, the curve will be flatter.

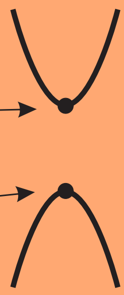
Parabolas are ***symmetrical***.



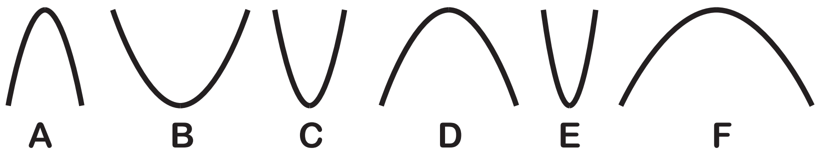
All parabolas have ***turning points***.

If the parabola is a 'cup' shape, it has a ***minimum*** turning point.

If the parabola is a 'hat' shape, it has a ***maximum*** turning point.



6. (a) Which of the following parabolas have a minimum turning point?
(b) Which of the following parabolas have a maximum turning point?



7. (a) Which of the following parabolas would have a minimum turning point if graphed?
(b) Which of the following parabolas would have a maximum turning point if graphed?
(c) Which of the following parabolas would be 'thinner' than the graph of $y = x^2$?
(d) Which of the following parabolas would be 'flatter' than the graph of $y = x^2$?

A $y = 3x^2$	**B** $y = x^2 - 5$
C $y = -4x^2$	**D** $y = \frac{1}{2}x^2$
E $y = 3x^2 - 2x - 5$	**F** $y = -2x^2 + 6x - 7$
G $y = -\frac{2}{3}x^2 - 5x + 1$	**H** $y = \frac{1}{3}x^2 - 5$
I $y = 3 + x^2$	**J** $y = 4 - 2x + \frac{3}{4}x^2$
K $y = 6 - 2x^2$	**L** $y = 5 - 6x - 3x^2$
M $y = 5 - 4x^2 + 2x$	**N** $y = -4x + \frac{2}{5}x^2$
O $y = -5 - \frac{1}{4}x + \frac{3}{8}x^2$	**P** $y = -9x^2 + 5x + 8$

Sketching Parabolas

Parabolas can be sketched knowing several of the following characteristics:

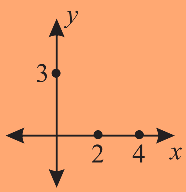
- the axis of symmetry
- the turning point
- x-intercepts (the points where the parabola crosses the x-axis)
- y-intercept (the point where the parabola crosses the y-axis)

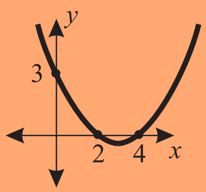
When sketching the parabolas:

- first mark the known points on a set of axes
- then connect them with a ***smooth*** and ***symmetrical*** curve.

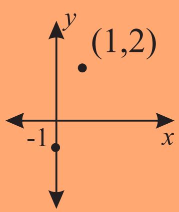
Examples Sketch the following parabolas

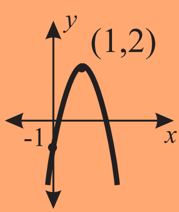
1. This parabola has:
- x-intercepts at 2 and 4
- y-intercept at 3



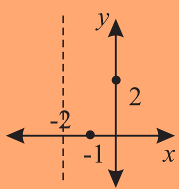


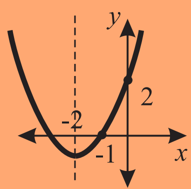
2. This parabola has:
- its turning point at (1,2)
- y-intercept at -1



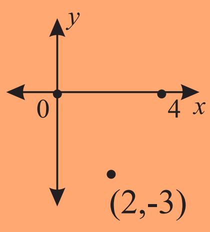


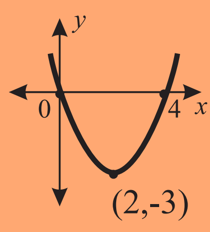
3. This parabola has:
- one x-intercept at -1
- y-intercept at 2
- axis of symmetry at $x = -2$





4. This parabola has:
- its turning point at (2,-3)
- x-intercepts at 0 and 4





EXERCISE 10B

Sketch the following parabolas.

1. This parabola has - x-intercepts at -2 and 2
   - y-intercept at -3
2. This parabola has - its turning point at (2,-1)
   - y-intercept at 4
3. This parabola has - one x-intercept at 2
   - y-intercept at -4
   - its axis of symmetry at $x = 3$
4. This parabola has - its turning point at (-1,4)
   - x-intercepts at -3 and 1
5. This parabola has - its axis of symmetry at $x = -3$
   - one x-intercept at -1
   - y-intercept at 1
6. This parabola has - x-intercepts at 1 and 5
   - y-intercept at -4
7. This parabola has - its turning point at (2,-1)
   - y-intercept at -4
8. This parabola has - x-intercepts at -5 and 1
   - y-intercept at -1
9. This parabola has - its turning point at (-2,3)
   - y-intercept at 5
10. This parabola has - its turning point at (4,0)
    - y-intercept at 1
11. This parabola has - its axis of symmetry at $x = 2$
    - one x-intercept at 5
    - y-intercept at -2
12. This parabola has - its axis of symmetry at $x = -3$
    - one x-intercept at 3
    - y-intercept at -2
13. This parabola has - its axis of symmetry at $x = 0$
    - one x-intercept at -1
    - y-intercept at -3
14. This parabola has - its axis of symmetry at $x = 0$
    - y-intercept at 0
    - passes through the point (2,4)

Finding x- and y-Intercepts

The y-intercept of a parabola can be found by letting $x = 0$
The x-intercepts of a parabola can be found by letting $y = 0$

Example 1 $y = x^2 - 5x + 6$

Step 1 Find the y-intercept by letting $x = 0$

$y = 0^2 - 5 \times 0 + 6$

$\mathbf{y = 6}$

Step 2 Find the x-intercepts by letting $y = 0$

$0 = x^2 - 5x + 6$

$x^2 - 5x + 6 = 0$

This can be solved by following the steps set out in Chapter 7 - the sections on solving quadratic equations (p 182).

$x^2 - 5x + 6 = 0$

$(x - 2)(x - 3) = 0$

There are two solutions:

$x - 2 = 0$ and $x - 3 = 0$

$\mathbf{x = 2}$ and $\mathbf{x = 3}$

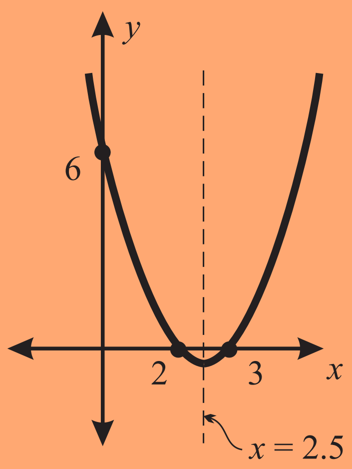
Step 3 The parabola can now be sketched:

y-intercept = 6

x-intercepts = 2 and 3

Note: the axis of symmetry will be midway between the x-intercepts.

In this example the axis of symmetry will be at $x = 2.5$



Example 2 $y = 6x^2 - x - 15$

Step 1 Find the y-intercept by letting $x = 0$

$\boldsymbol{y = -15}$

Step 2 Find the x-intercepts by letting $y = 0$

$0 = 6x^2 - x - 15$

$6x^2 - x - 15 = 0$

$(2x + 3)(3x - 5) = 0$

There are two solutions:

$2x + 3 = 0$ and $3x - 5 = 0$

$\boldsymbol{x = -\frac{3}{2}}$ and $\boldsymbol{x = \frac{5}{3}}$

Step 3 The parabola can now be sketched:

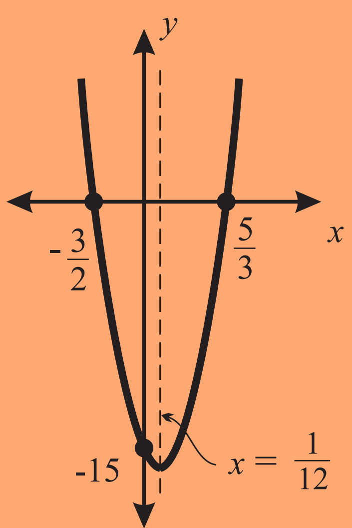
y-intercept $= -15$

x-intercepts $= -\frac{3}{2}$ and $\frac{5}{3}$

Note: the axis of symmetry will be midway between the x-intercepts.

In this example the axis of symmetry will be at

$$x = \frac{-\frac{3}{2} + \frac{5}{3}}{2} = \frac{1}{12}$$



Any quadratic equation can be solved using the quadratic formula (p185) to find the x-intercepts.

Example 3 $y = -8x^2 - 6x + 35$

Step 1 Find the y-intercept by letting $x = 0$

$\mathbf{y = 35}$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Step 2 Find the x-intercepts by letting $y = 0$ and using the quadratic formula.

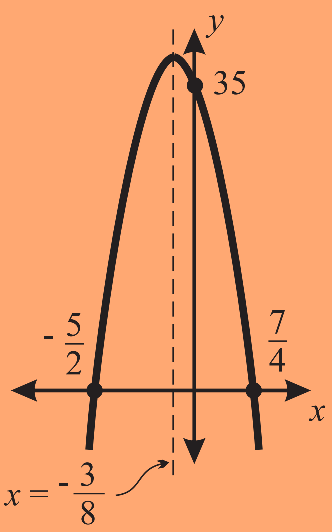
$a = -8$, $b = -6$ and $c = 35$.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times -8 \times 35}}{2 \times -8}$$

$$x = \frac{6 \pm 34}{-16} = -\frac{5}{2} \text{ and } \frac{7}{4}$$

Step 3 Axis of symmetry is at

$$x = \frac{-\frac{5}{2} + \frac{7}{4}}{2} = -\frac{3}{8}$$



EXERCISE 10C

1. Find the x- and y-intercepts of the graphs for the following equations and sketch the graphs showing all intercepts and the axis of symmetry.

(a) $y = x^2 + 4x - 32$
(b) $y = x^2 - 14x + 45$
(c) $y = x^2 + 7x + 10$
(d) $y = -x^2 - 4x + 12$
(e) $y = 2x^2 - 19x + 42$
(f) $y = -4x^2 - 7x + 15$
(g) $y = 6x^2 + 11x - 35$
(h) $y = -12x^2 - x + 20$

2. Choose several of the equations from question 1 and plot these on a graphics calculator. The scales may need to be changed to view each graph clearly.

Finding the Turning Point Using the Axis of Symmetry

The turning point of a parabola can be found by firstly finding the axis of symmetry then substituting this x-value into the quadratic equation.

Example Find the turning point of the parabola with equation

$$y = x^2 - 4x - 5$$

Step 1 Find the x-intercepts.

$$0 = x^2 - 4x - 5$$
$$0 = (x - 5)(x + 1)$$

x-intercepts are $x = 5$ and -1

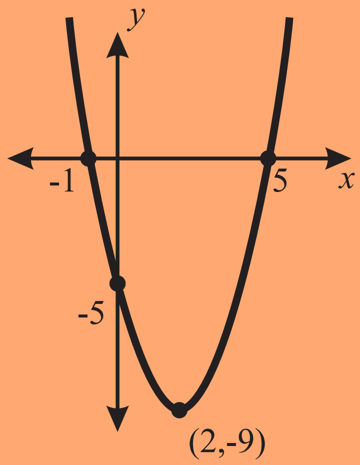
Step 2 Find the axis of symmetry (midway between the x-intercepts)

$$x = \frac{5 + -1}{2}$$
$$x = 2$$

Step 3 Substitute this into the equation to find the y-value of the turning point

$$y = x^2 - 4x - 5$$
$$= 4 - 8 - 5$$
$$y = -9$$

Step 4 The coordinates of the turning point are (2,-9)



EXERCISE 10D

Find the coordinates of the turning point for each of the parabolas from question 1 in exercise 10C.

Finding the Turning Point Using $y = a(x - h)^2 + k$

Using the axis of symmetry to find the turning point can only be used when the parabola has x-intercepts. The turning point of any parabola can be found by converting the quadratic equation to the form:

$\boxed{y = a(x - h)^2 + k}$ This is called ***turning point form.***

The coordinates of the turning point are (h,k).

Example 1 $y = x^2 - 6x - 2$

To convert to ***turning point form*** it is necessary to ***complete the square*** of the first two terms (refer to page 156).

$$\begin{aligned} y &= x^2 - 6x - 2 \\ &= x^2 - 6x + 9 - 9 - 2 \\ y &= (x - 3)^2 - 11 \end{aligned}$$

Turning point is **(3,-11)**

Example 2

$$\begin{aligned} y &= x^2 + 8x + 3 \\ &= x^2 + 8x + 16 - 16 + 3 \\ &= (x + 4)^2 - 13 \\ y &= [x - (-4)]^2 - 13 \end{aligned}$$

Turning point is **(-4,-13)**

Example 3

$$\begin{aligned} y &= x^2 + 3x + 8 \\ &= x^2 + 3x + (\tfrac{3}{2})^2 - (\tfrac{3}{2})^2 + 8 \\ &= x^2 + 3x + \tfrac{9}{4} - \tfrac{9}{4} + 8 \\ &= (x + \tfrac{3}{2})^2 + \tfrac{23}{4} \\ y &= [x - (-\tfrac{3}{2})]^2 + \tfrac{23}{4} \end{aligned}$$

Turning point is $\left(-\frac{3}{2}, \frac{23}{4}\right)$

Example 4

$$\begin{aligned} y &= -x^2 - 12x - 23 \\ &= -(x^2 + 12x + 23) \\ &= -(x^2 + 12x + 36 - 36 + 23) \\ &= -[(x+6)^2 - 13] \\ &= -(x+6)^2 + 13 \\ &= -[x - (-6)]^2 + 13 \end{aligned}$$

Turning point is **(-6,13)**

Example 5

$$\begin{aligned} y &= -4x^2 + 24x + 36 \\ &= -4(x^2 - 6x - 9) \\ &= -4(x^2 - 6x + 9 - 9 - 9) \\ &= -4[(x-3)^2 - 18] \\ &= -4(x-3)^2 + 72 \end{aligned}$$

Turning point is **(3,72)**

EXERCISE 10E

1. Find the coordinates of the turning point for the parabolas with the following equations.

(a) $y = x^2 - 10x + 2$ (b) $y = x^2 - 16x + 24$
(c) $y = x^2 + 4x - 3$ (d) $y = x^2 + 6x + 5$
(e) $y = x^2 + 2x - 8$ (f) $y = x^2 - 14x + 1$
(g) $y = x^2 - 20x$ (h) $y = x^2 + 18x$

2. Find the coordinates of the turning point for the parabolas with the following equations.

(a) $y = x^2 - 5x + 3$ (b) $y = x^2 - x + 4$
(c) $y = x^2 + 3x - 8$ (d) $y = x^2 + 9x - 3$

3. Find the coordinates of the turning point for the parabolas with the following equations.

(a) $y = -x^2 + 4x - 5$ (b) $y = -x^2 - 12x - 3$
(c) $y = -2x^2 + 8x - 16$ (d) $y = -3x^2 - 12x + 15$
(e) $y = -2x^2 - 5x + 8$ (f) $y = -4x^2 + 6x - 12$
(g) $y = -5x^2 + 15x - 10$ (h) $y = -6x^2 - 8x + 15$

Turning Point Form

The y-intercept of a parabola can be found by letting $x = 0$.

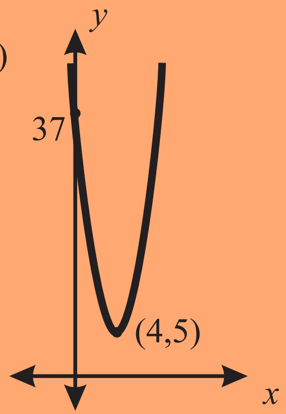
Example 1 Find the turning point and y-intercept of the following equation and sketch the parabola.

$$y = 2(x - 4)^2 + 5$$

The turning point is (4,5) (Exercise 10E)

The y-intercept is found by letting $x = 0$

$$\begin{aligned} y &= 2(0 - 4)^2 + 5 \\ &= 2 \times (-4)^2 + 5 \\ &= 2 \times 16 + 5 \\ \boldsymbol{y} &\boldsymbol{= 37} \end{aligned}$$



Remember the general equation of a parabola is $y = a(x - h)^2 + k$
If $\boldsymbol{a}$ is ***positive*** the parabola is a '***cup***'.
If $\boldsymbol{a}$ is ***negative*** the parabola is a '***hat***'.

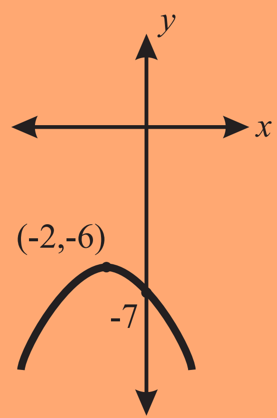
Example 2 Find the turning point and y-intercept of the following equation and sketch the parabola.

$$y = -\tfrac{1}{4}(x + 2)^2 - 6$$ (This parabola will be a 'hat' because a is negative)

The turning point is (-2,-6) (Exercise 10E)

The y-intercept is found by letting $x = 0$

$$\begin{aligned} y &= -\tfrac{1}{4}(0 + 2)^2 - 6 \\ &= -\tfrac{1}{4} \times (2)^2 - 6 \\ &= -\tfrac{1}{4} \times 4 - 6 \\ \boldsymbol{y} &\boldsymbol{= -7} \end{aligned}$$



EXERCISE 10F

1. Sketch the following parabolas by locating the turning point and y-intercept of each.

(a) $y = 2(x - 3)^2 - 4$ (b) $y = \frac{1}{2}(x - 4)^2 + 1$

(c) $y = -2(x + 1)^2 - 2$ (d) $y = 4(x - 3)^2$

(e) $y = -3(x - 5)^2 - 6$ (f) $y = \frac{1}{3}(x + 3)^2 + 4$

(g) $y = -\frac{1}{4}(x - 8)^2 + 3$ (h) $y = -(x + 4)^2 + 1$

2. By analysing these graphs, which of the parabolas have x-intercepts?

Practical Applications

Many shapes that occur naturally and man made are in the shape of parabolas. For example: a piece of string joined to two points, the motion of a ball through the air, a bent piece of plastic, a stream of water spurting from a hose, many styles of bridges, radar dishes, etc.

Example

A golf ball is hit and travels 200 metres before it hits the ground. It just passes over a 20 metre tall tree at the top of its trajectory. Assume that the trajectory of the golf ball is a parabola.

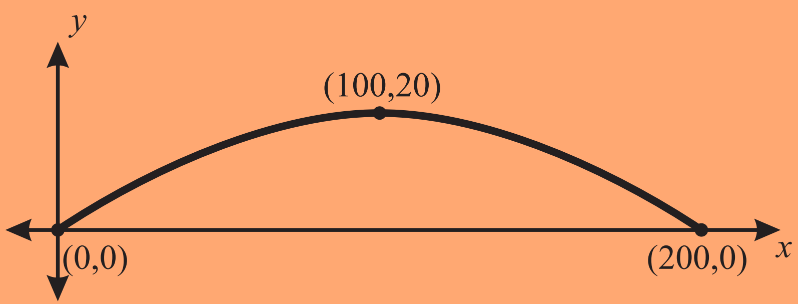
1. Sketch the parabolic trajectory of the ball on coordinate axes using the point that it was hit as the origin. State the coordinates of the turning point and x-intercepts.

2. Find the equation of this parabola.

3. Find the height of the ball 50 metres from the point it was hit.

Answers

1.



2. The turning point form of the equation to a parabola is:

$$y = a(x - h)^2 + k$$

The coordinates of the turning point are (h,k).
The turning point is (100,20). Substitute these into the equation:

$$y = a(x - 100)^2 + 20$$

To find a, substitute one of the points into this equation.
The easiest point is (0,0).

$$0 = a(0 - 100)^2 + 20$$
$$0 = 10\,000a + 20$$
$$a = -\frac{20}{10\,000}$$
$$= -\frac{1}{500}$$

Complete the equation:

$$\boxed{y = -\frac{1}{500}(x - 100)^2 + 20}$$

Example continued

$$y = -\frac{1}{500}(x - 100)^2 + 20$$

3. Find the height of the ball, y, when $x = 50$.

$$y = -\frac{1}{500}(50 - 100)^2 + 20$$
$$= -\frac{1}{500}(-50)^2 + 20$$
$$= -5 + 20$$
$$\mathbf{y = 15 \text{ metres}}$$

Note: the trajectories of objects through air in reality are not perfectly symmetrical parabolas due to spin and air resistance. They are often assumed to be parabolic to make the mathematics simpler.

Assume that all the trajectories in the following problems are parabolic.

EXERCISE 10G

1. A golf ball is hit and travels 100 metres before it hits the ground. It is estimated that the maximum height reached by the ball is 25 metres.
(a) Sketch the trajectory of the ball on coordinate axes using the origin as the point the ball was hit.
(b) Find the equation of this parabola.
(c) Find the height of the ball 20 metres from where it was hit.

2. A footballer kicks the ball from 20 metres out from the goal. The ball just clears the 12 metre high goal post at the top of its trajectory.
(a) Sketch the trajectory of the ball on coordinate axes using the origin as the point the ball was kicked.
(b) Find the equation of this parabola.
(c) Find the height of the ball 5 metres from where it was kicked.

3. The trajectory of two hits in cricket were analysed and the equation for each trajectory is shown below.

Hit **A**: $y = -\frac{2}{125}(x - 50)^2 + 40$

Hit **B**: $y = -\frac{1}{120}(x - 60)^2 + 30$

For each hit find:

(a) the distance travelled by the ball

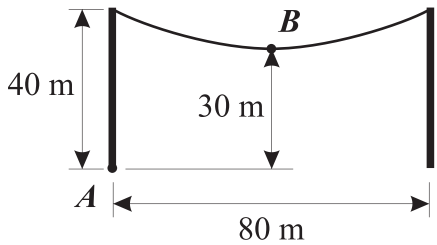
(b) the maximum height reached by the ball

(c) the height of the ball above a fielder standing 30 metres from the batter.

4. A cable hangs between two pylons as shown in this diagram. The pylons are 40 metres high and 80 metres apart. The lowest point of the cable is 30 metres above the ground.

(a) Using point A as the origin, find the equation of the parabolic shape of the cable.

(b) Copy and complete the table below that states the height of the cable every 10 metres from point A. Give answers correct to one decimal place.



Distance from A (m)	0	10	20	30	40	50	60	70	80
Height above ground (m)									

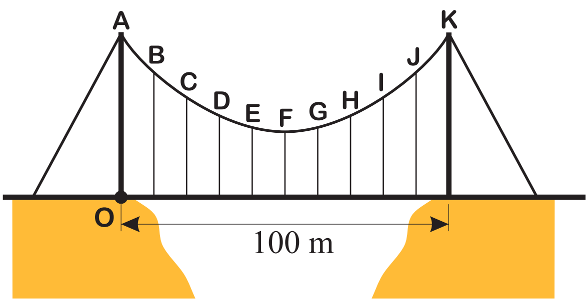
5. The equation for the parabolic cable on this bridge is:

$$y = \frac{1}{100}(x - 50)^2 + 25$$

Point **O** is the origin.

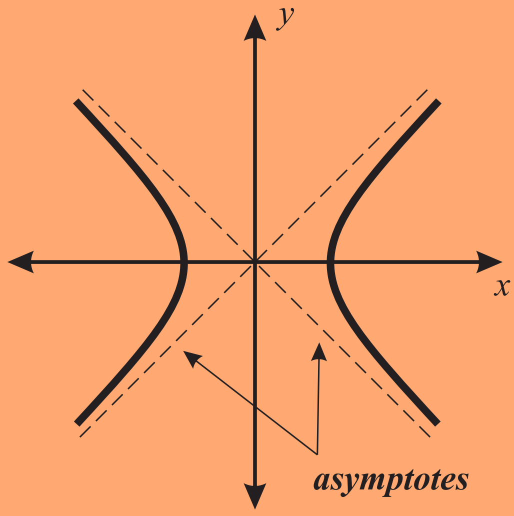
(a) Find the height of the pylons **A** and **K**.

(b) Find the lengths of the equally spaced supporting cables (**B** - **J**).

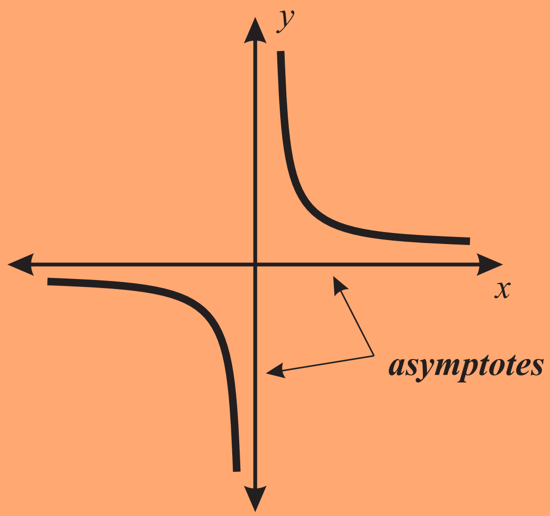


Rectangular Hyperbolas

Hyperbolas (hyperbolae) are a family of curves that approach but never reach lines called asymptotes. An example of a hyperbola is shown below. The asymptotes are shown by dotted lines.



The mathematics of the general shape of these curves will be studied in detail in later year levels.
We will study now hyperbolas that have asymptotes that are the x-axis and y-axis. They are called ***rectangular hyperbolas*** because the angle between the asymptotes is 90°.



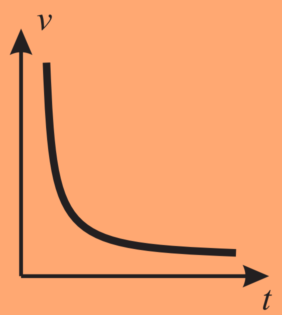
The general relationship for a rectangular hyperbola is:

$$xy = a \qquad \text{or} \qquad y = \frac{a}{x}$$

This is the relationship in many practical situations. In practical situations only the positive part of the graph applies.

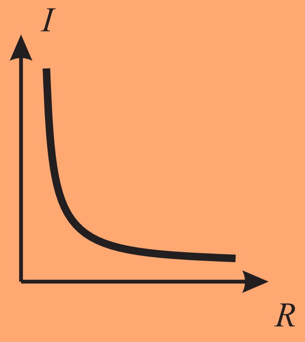
Examples

The relationship between velocity (***v***) and time (***t***) for an object travelling over a set distance (***d***).



$$v = \frac{d}{t}$$

The relationship between current (***I***) and resistance (***R***) is a simple electrical circuit where there is a set voltage (***V***).



$$I = \frac{V}{R}$$

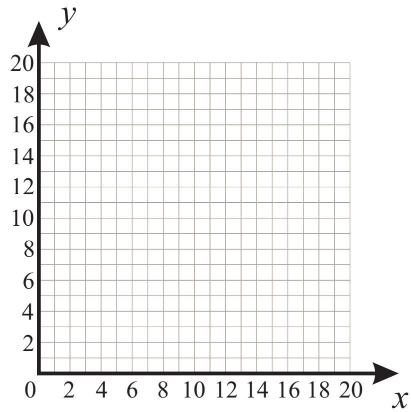
EXERCISE 10H

1. (a) Copy and complete the table of values for following relationship.

$$y = \frac{20}{x}$$

x	1	2	4	5	10	20
y						

(b) Copy and complete this graph by plotting the points from the above table. Connect the points with a smooth curve.



2. The tables of values below is completed for the given equation.

$$y = \frac{a}{x}$$

x	1	2	4	5	10	20	50
y	100	50	25	20	10	5	2

(a) Find a.

(b) Find y for the following values of x.

(i) 25 (ii) 200 (iii) 0.5 (iv) $\frac{1}{4}$

3. The tables of values below is completed for the given equation where P is a constant.

$$M = \frac{P}{Q}$$

Q	0.01	0.5	0.1	1	10	50	100
M	250	5	25	2.5	0.25	0.05	0.025

(a) Find P.

(b) Find M for the following values of Q.

(i) 50 (ii) 0.5 (iii) 0.1 (iv) 0.025

4. The current (***I***), in amps, is measured in an electric circuit for different resistances (***R***), measured in ohms, and shown in the table below.

I	1	2	4	6	8	12	24
R	24	12	6	4	3	2	1

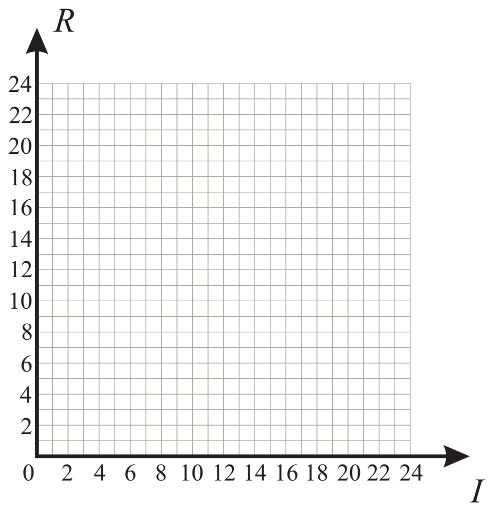
(a) The relationship between current and resistance is known to be of the form shown here.
Find V.

$$R = \frac{V}{I}$$

(b) Find ***R*** for the following currents:

(i) 0.1 (ii) 0.2 (iii) 1.2 (iv) 48 (v) 240

(c) Copy and complete this graph by plotting the points from the above table.
Connect the points with a smooth curve.

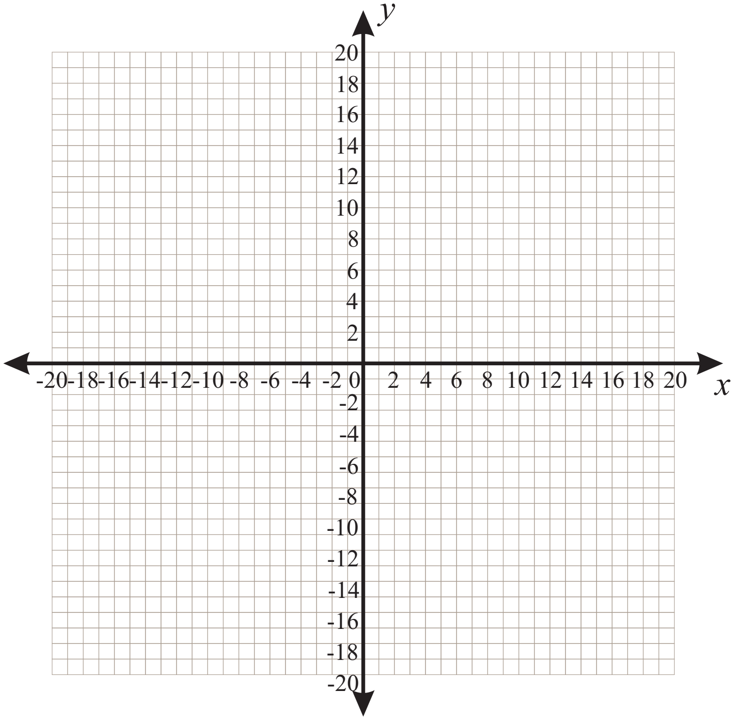


5. (a) Copy and complete the table of values below for this relationship.

$$y = \frac{10}{x}$$

x	-20	-10	-5	-2	-1	-0.5	0.5	1	2	5	10	20
y												

(b) Copy and complete the graph below by plotting the points from the above table. Connect the points with smooth curves.



6. (a) Copy and complete the table of values below for this relationship.

$$y = \frac{20}{x}$$

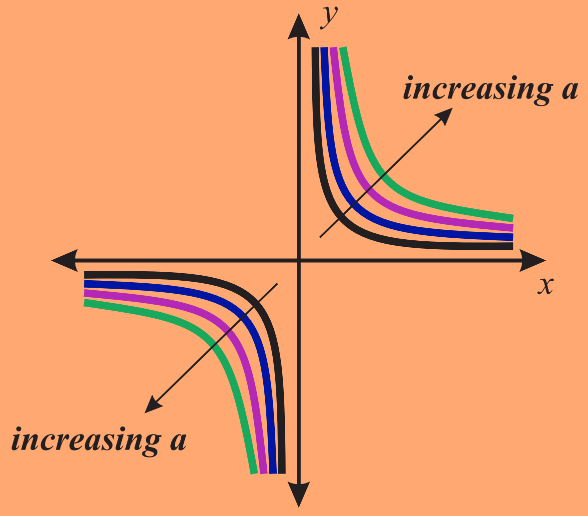
x	-20	-10	-5	-4	-2	-1	1	2	4	5	10	20
y												

(b) Plot these points on the same graph as in question 5 and connect with smooth curves.

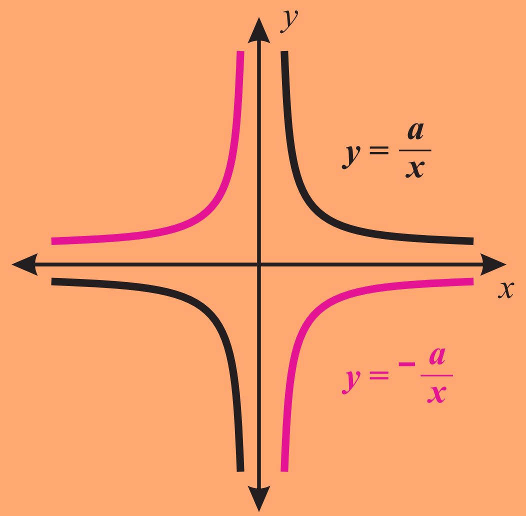
7. (a) Copy the table of values from question 5 and complete for this relationship.
(b) Plot on the same graph and connect with smooth curves.

$$y = \frac{-10}{x}$$

As seen in questions 5 & 6, changing the value of $\boldsymbol{a}$ in the general equation of a hyperbola $y = \frac{a}{x}$ changes the shape of the curves. The diagram below shows the effect of increasing the value of $\boldsymbol{a}$ on the curve.



It can also be seen from question 7 that a negative sign reflects the hyperbola about the axes.



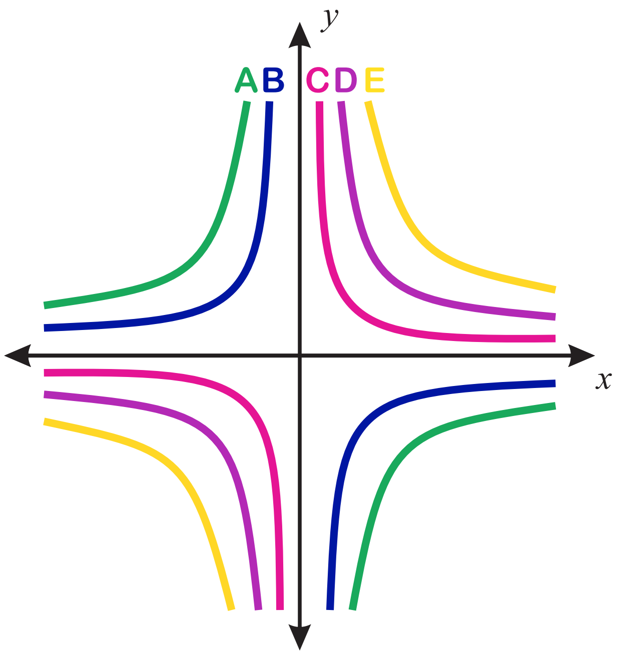
EXERCISE 10I

1. Match the graphs below with the following relationships.

(a) $y = \frac{a}{x}$ (b) $y = -\frac{2a}{x}$

(c) $y = \frac{a}{3x}$ (d) $y = \frac{5a}{x}$

(e) $y = -\frac{a}{2x}$



Note:

$$y = \frac{\frac{1}{3}a}{x}$$

can be written as

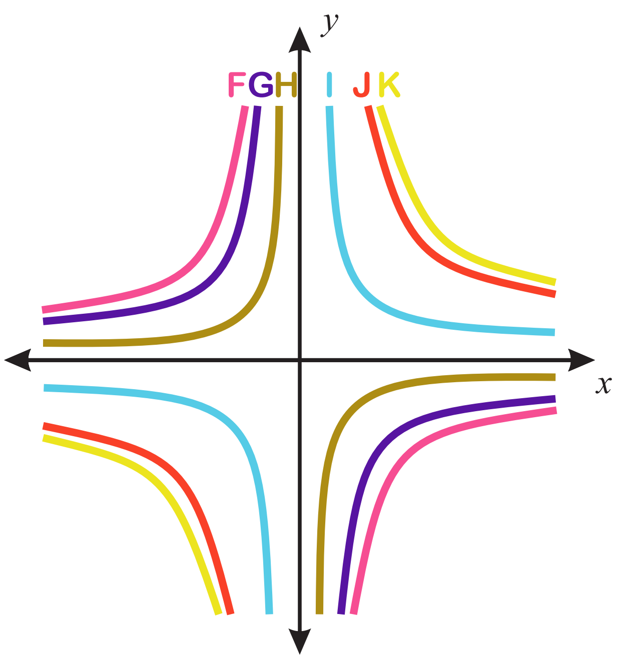
$$y = \frac{a}{3x}$$

2. Match the graphs below with the following relationships.

(a) $y = \frac{5}{x}$ (b) $y = -\frac{3}{x}$

(c) $y = \frac{1}{4x}$ (d) $y = \frac{7}{x}$

(e) $y = -\frac{1}{6x}$ (f) $y = -\frac{1}{2x}$

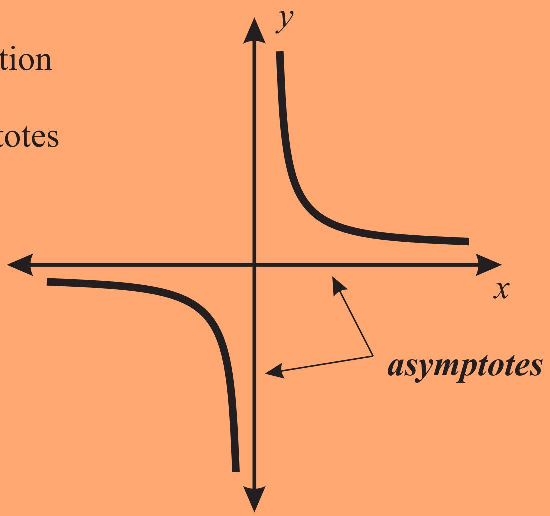


Transforming Hyperbolas

So far we have only considered hyperbolas that have asymptotes that intersect at the origin.

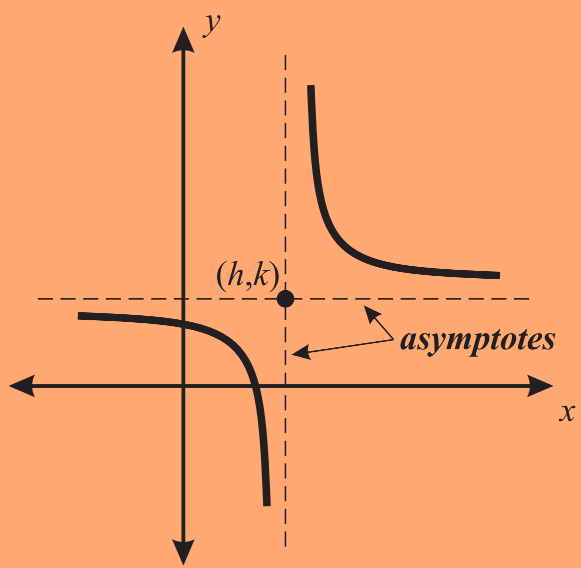
As noted earlier, the equation of a hyperbola with the intersection of the asymptotes at the origin is:

$$y = \frac{a}{x}$$



If a hyperbola is moved so that the intersection of the asymptotes is at the point (h,k) then the equation of the hyperbola will be:

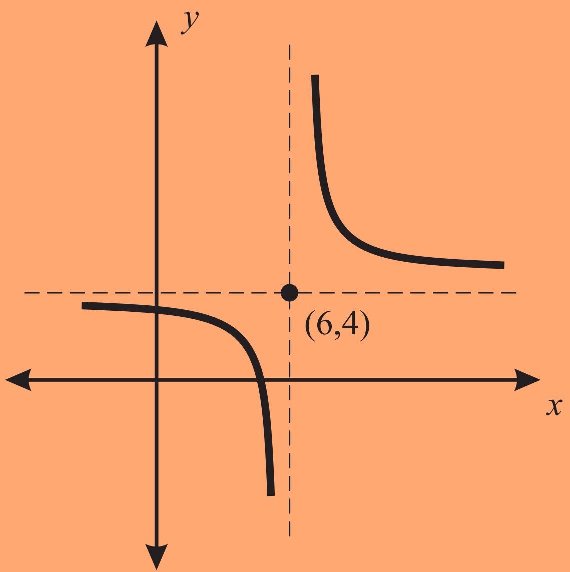
$$y = \frac{a}{(x - h)} + k$$



Example 1

The hyperbola with equation $\mathbf{y = \frac{2}{x}}$ has been moved so that the point of intersection of the asymptotes is (6,4) as shown on the graph below.

Write the equation of the hyperbola.



Answer

Substitute the values for h and k into the equation of the hyperbola.

$$y = \frac{a}{(x - h)} + k$$

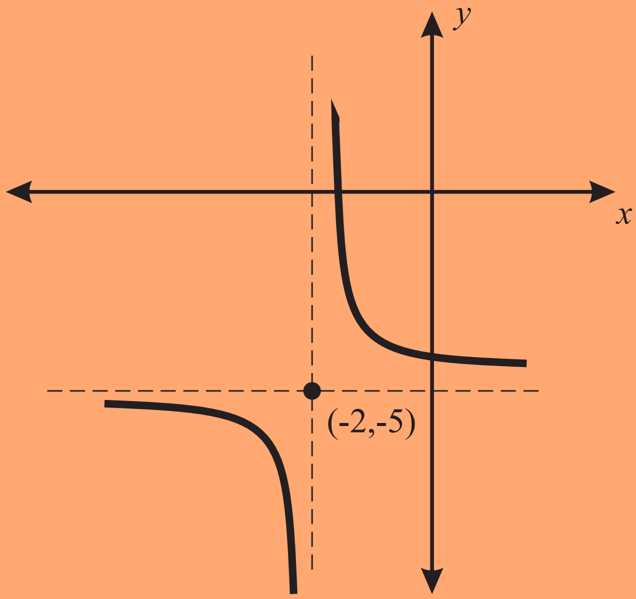
$$h = 6$$
$$k = 4$$

$$\mathbf{y = \frac{2}{(x - 6)} + 4}$$

Example 2

The hyperbola with equation $\boldsymbol{y = \frac{3}{x}}$ has been moved so that the point of intersection of the asymptotes is (-2,-5) as shown on the graph below.

Write the equation of the hyperbola.



Answer

Substitute the values for h and k into the equation of the hyperbola.

$$y = \frac{a}{(x - h)} + k$$

$$h = -2$$
$$k = -5$$

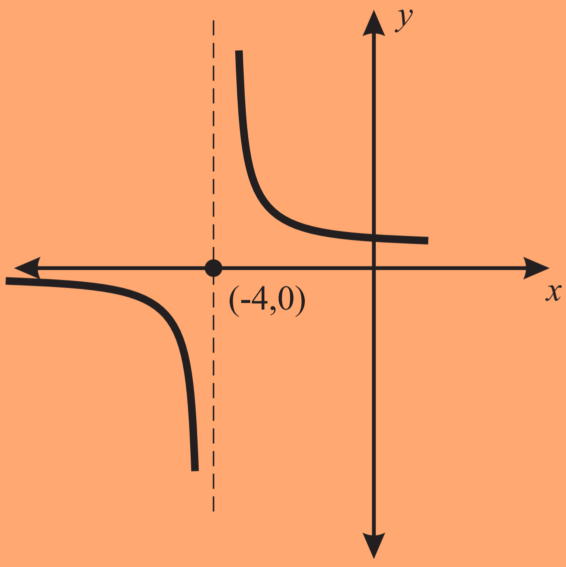
$$y = \frac{3}{(x - -2)} - 5$$

$$\boldsymbol{y = \frac{3}{(x + 2)} - 5}$$

Example 3

The hyperbola with equation $\mathbf{y = \frac{1}{x}}$ has been moved so that the point of intersection of the asymptotes is (-4,0) as shown on the graph below.

Write the equation of the hyperbola.



Answer

Substitute the values for h and k into the equation of the hyperbola.

$$y = \frac{a}{(x - h)} + k$$

$$h = -4$$
$$k = 0$$

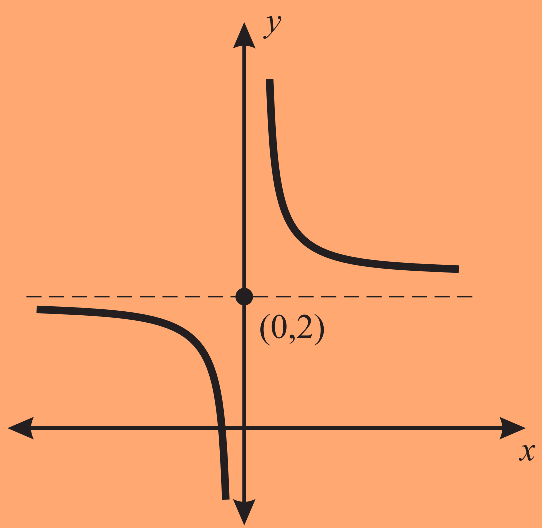
$$y = \frac{1}{(x - -4)} + 0$$

$$\mathbf{y = \frac{1}{(x + 4)}}$$

Example 4

The hyperbola with equation $\boldsymbol{y = \frac{4}{x}}$ has been moved so that the point of intersection of the asymptotes is (0,2) as shown on the graph below.

Write the equation of the hyperbola.



Answer

Substitute the values for h and k into the equation of the hyperbola.

$$y = \frac{a}{(x - h)} + k$$

$$h = 0$$
$$k = 2$$

$$y = \frac{4}{(x - 0)} + 2$$

$$\boldsymbol{y = \frac{4}{x} + 2}$$

EXERCISE 10J

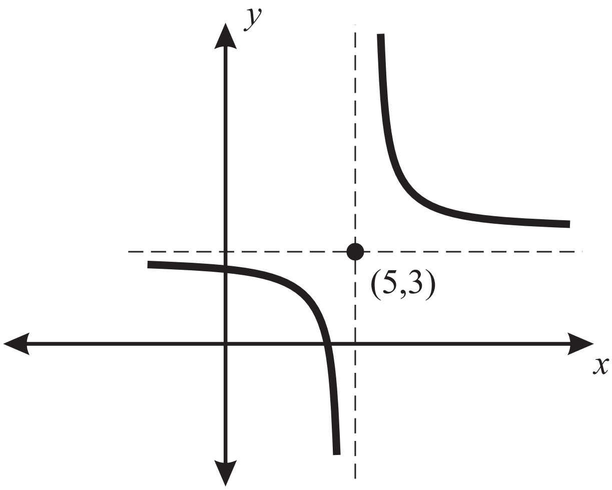
1. Which of the equations below (**A** - **D**) could be the equation of the hyperbola shown.

A $y = \dfrac{2}{(x+5)} + 3$

B $y = \dfrac{2}{(x-3)} + 5$

C $y = \dfrac{2}{(x-5)} - 3$

D $y = \dfrac{2}{(x-5)} + 3$



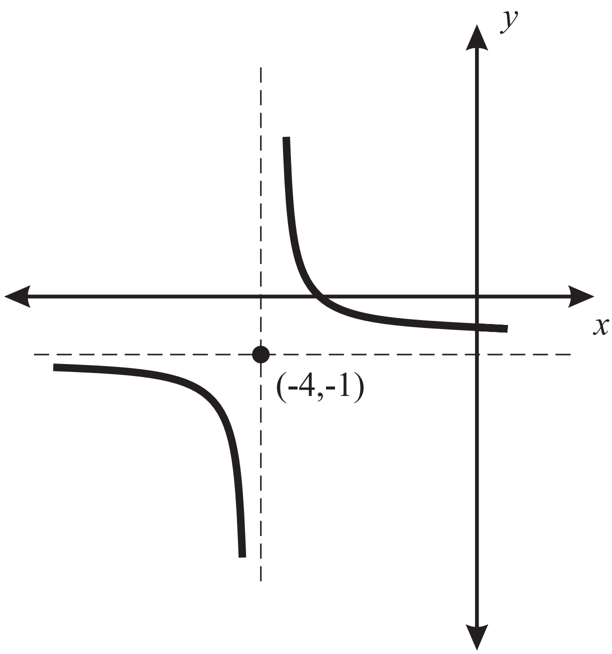
2. Which of the equations below (**A** - **D**) could be the equation of the hyperbola shown.

A $y = \dfrac{3}{(x-4)} + 1$

B $y = \dfrac{3}{(x+1)} - 4$

C $y = \dfrac{3}{(x+4)} - 1$

D $y = \dfrac{3}{(x+1)} - 4$



3. Which of the following equations (**A** - **D**) could be the equation of a hyperbola that has the intersection of its asymptotes at the point (5,-2).

A $y = \dfrac{1}{(x - 5)} + 2$

B $y = \dfrac{2}{(x - 2)} + 5$

C $y = \dfrac{3}{(x - 5)} - 2$

D $y = \dfrac{4}{(x + 5)} - 2$

4. Which of the following equations (**A** - **D**) could be the equation of a hyperbola that has the intersection of its asymptotes at the point (0,4).

A $y = \dfrac{5}{(x - 4)}$

B $y = \dfrac{6}{x} + 4$

C $y = \dfrac{1}{x} - 4$

D $y = \dfrac{2}{(x + 4)}$

5. The hyperbola with equation $\boldsymbol{y = \dfrac{1}{x}}$ is transposed so that the intersection of its asymptotes is at the points with the following coordinates. Write the equation of the transposed hyperbola for each point.

(a) (3,7) (b) (-5,0) (c) (-1,-5) (d) (0,8) (e) (6,-5)

6. Write the coordinates of the point where the asymptotes intersect for the hyperbolas with following equations and sketch each hyperbola.

(a) $y = \dfrac{3}{(x - 9)} - 1$

(b) $y = \dfrac{5}{(x + 2)} + 7$

(c) $y = \dfrac{1}{(x - 8)}$

(d) $y = \dfrac{1}{x} + 4$

(e) $y = \dfrac{3}{(x + 1)} - 6$

(f) $y = \dfrac{9}{(x + 8)} - 5$

(g) $y = \dfrac{12}{(x + 2)}$

(h) $y = \dfrac{6}{x} - 5$

Circles

The equation of a circle with radius $\boldsymbol{r}$ and centre at the origin of coordinate axes is:

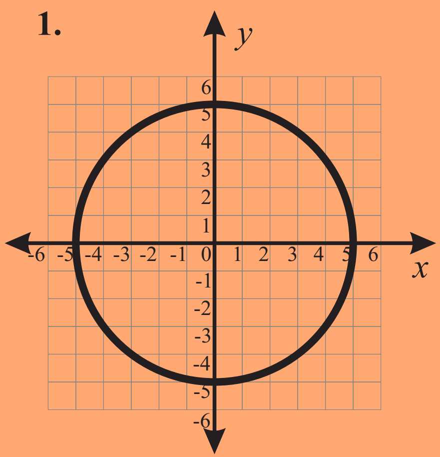
$$x^2 + y^2 = r^2$$

If the centre of the circle is at coordinates (h,k) then the equation of the circle is:

$$(x - h)^2 + (y - k)^2 = r^2$$

Examples

1.

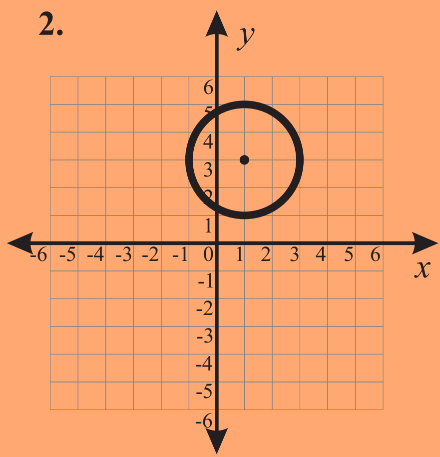


Centre at the origin
Radius = 5

$$x^2 + y^2 = 5^2$$

$$\mathbf{x^2 + y^2 = 25}$$

2.

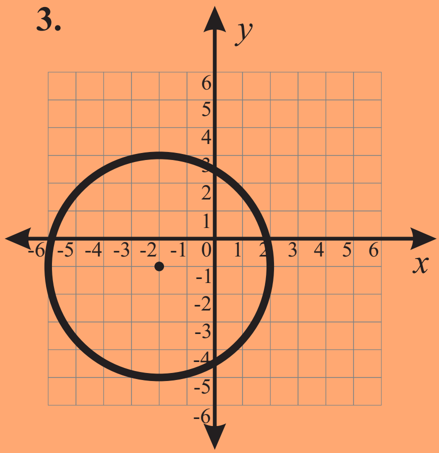


Centre at (1,3)
$h = 1$, $k = 3$
Radius = 2

$$(x - 1)^2 + (y - 3)^2 = 2^2$$

$$\mathbf{(x - 1)^2 + (y - 3)^2 = 4}$$

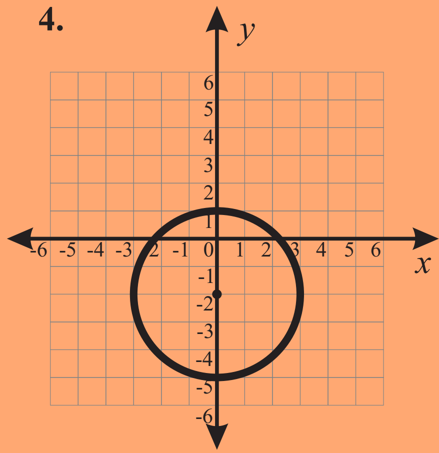
3.



Centre at (-2,-1)
$h = -2$, $k = -1$
Radius = 4
$(x - -2)^2 + (y - -1)^2 = 4^2$

$\mathbf{(x + 2)^2 + (y + 1)^2 = 16}$

4.



Centre at (0,-2)
$h = 0$, $k = -2$
Radius = 3
$(x - 0)^2 + (y - -2)^2 = 3^2$

$\mathbf{x^2 + (y + 2)^2 = 9}$

EXERCISE 10K

1. Write the equation of the circles with the following centres and radii.

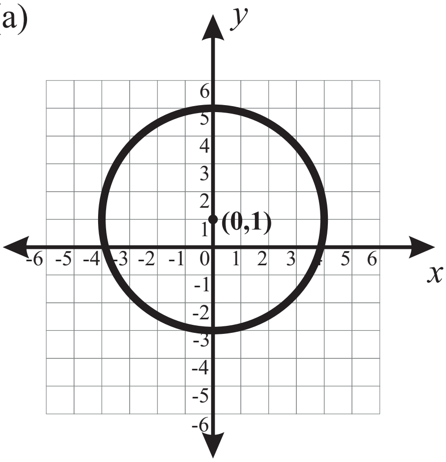
(a) Centre (0,0), radius = 5
(b) Centre (0,3), radius = 4
(c) Centre (4,0), radius = 1
(d) Centre (-5,0), radius = 3
(e) Centre (2,4), radius = 6
(f) Centre (-3,5), radius = 4
(g) Centre (6,-5), radius = 7
(h) Centre (7,9), radius = 12
(i) Centre (-5,-8), radius = 10
(j) Centre (-6,-1), radius = 9

2. Write the coordinates of the centre and the radius of the circles with the following equations.

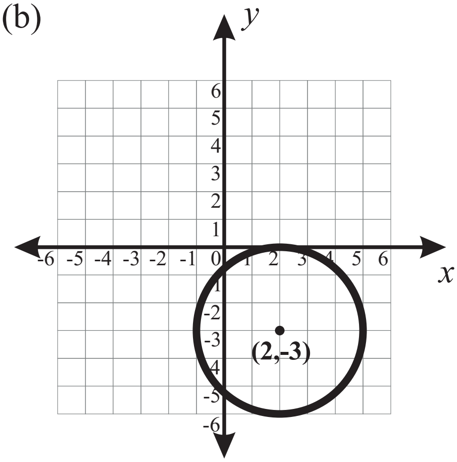
(a) $x^2 + y^2 = 64$
(b) $(x - 7)^2 + y^2 = 1$
(c) $(x + 3)^2 + (y - 5)^2 = 49$
(d) $x^2 + (y - 2)^2 = 81$
(e) $(x - 6)^2 + (y - 1)^2 = 36$
(f) $(x + 2)^2 + (y + 9)^2 = 100$
(g) $(x - 12)^2 + (y + 15)^2 = 144$
(h) $(x + 13)^2 + (y - 11)^2 = 169$

3. Write the equation of the circles shown below.

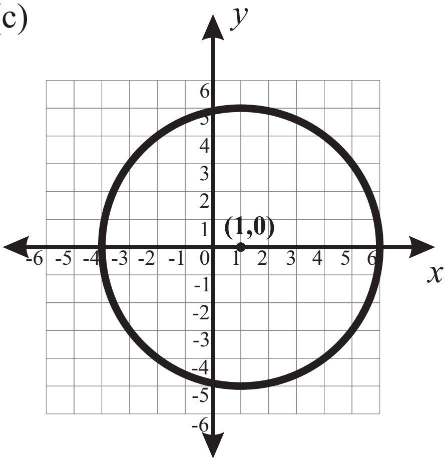
(a)



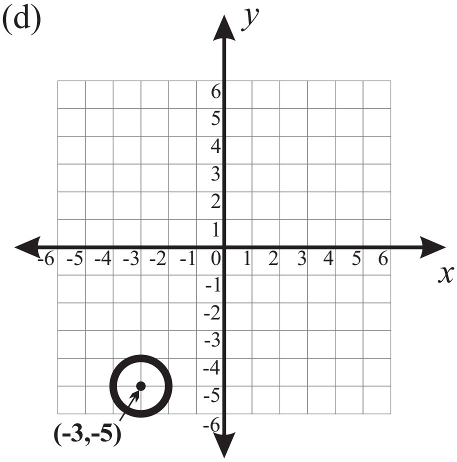
(b)



(c)



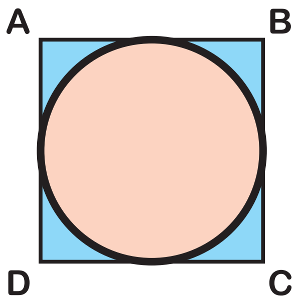
(d)



4. Find the equation to a circle that just fits in the square **ABCD**. The coordinates of the corners of the square are:

A = (-2,4) **B** = (8,4)

C = (8,-6) **D** = (-2,-6)



Sometimes the equation of a circle can be given in the form shown in the example below.

$$x^2 + 4x + y^2 - 6y = 14$$

The equation needs to be changed to find the coordinates of the centre and the radius. This is achieved by completing the square as shown on page 156.

Using this example follow the steps shown below.

Step 1 Complete the square for x.

$$x^2 + 4x + y^2 - 6y = 3$$

$$x^2 + 4x + \mathbf{4} + y^2 - 6y = 3 + \mathbf{4}$$

$$(x + 2)^2 + y^2 - 6y = 7$$

Remember if a number is added to one side of the equation the same number needs to added to the other side for the two sides of the equation to remain equal.

Step 2 Complete the square for y.

$$(x + 2)^2 + y^2 - 6y + \mathbf{9} = 7 + \mathbf{9}$$

$$(x + 2)^2 + (y - 3)^2 = 16$$

Step 3 Write the coordinates of the centre of the circle and its radius.

Coordinates of centre = (-2,3)
Radius = 4

EXERCISE 10L

Find the coordinates of the centre and the radius of the circles with the following equations.

1. $x^2 - 2x + y^2 - 10y = 10$

2. $x^2 - 6x + y^2 + 20y = -9$

3. $x^2 + y^2 - 12y = -35$

4. $x^2 + 8x + y^2 = 0$

5. $x^2 - 12x + y^2 + 6y = 4$

6. $x^2 - 24x + y^2 + 10y = 0$

7. $x^2 + 14x + y^2 - 16y = -13$

8. $x^2 - 20x + y^2 - 22y = 4$

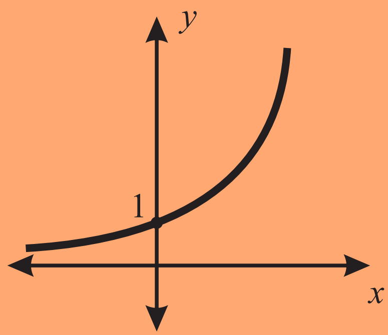
Exponentials

Practical applications of exponential graphs have been studied in detail on pages 95-100.

The basic mathematical equation for an exponential relationship is:

$$y = b^x$$

The shape of the graph of this relationship is:



Key features of this graph are:

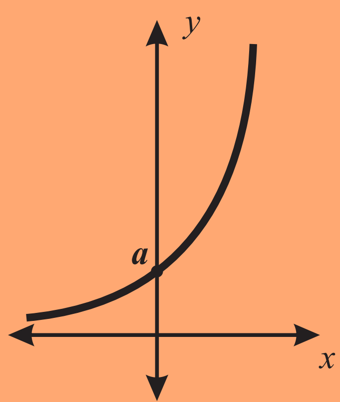
- the y-intercept is 1
- the x-axis is an asymptote

This basic graph can undergo one or more transformations that will change the position and/or the shape of the graph.

1. Multiplied by a constant $\boldsymbol{a}$.

$$y = a \times b^x$$

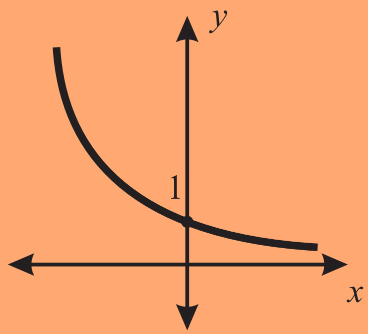
The graph is moved up and stretched.



2. Negative x

$$y = b^{-x}$$

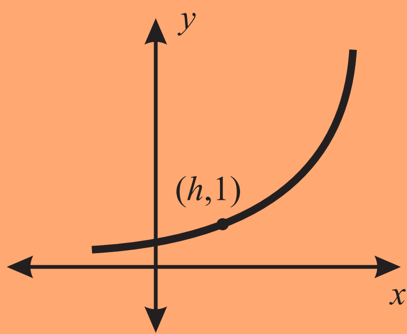
The graph is reflected about the y-axis.



3. Translated right or left by h.

$$y = b^{(x-h)}$$

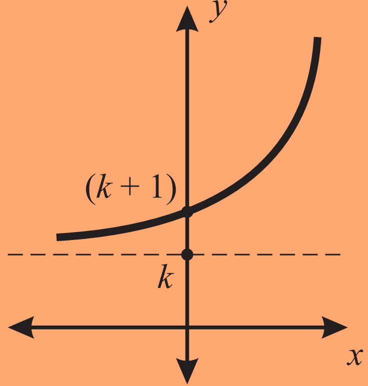
The point $(h,1)$ is a point on the graph.
The x-axis is still the asymptote.



4. Translated up or down by k.

$$y = b^{x} + k$$

$y = k$ is the asymptote



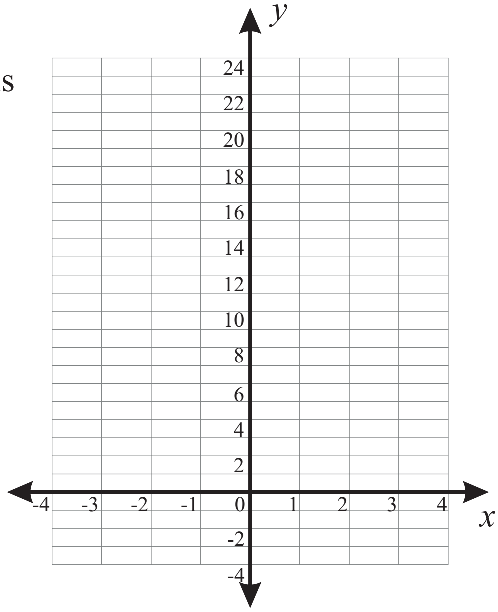
EXERCISE 10M

1. (a) Copy and complete the table of values below for the following relationship.

$$y = 2^x$$

x	-4	-3	-2	-1	0	1	2	3	4
y									

(b) Draw a set of axes with the x-axis from -4 to 4 and the y-axis from -4 to 24 as shown here.



(c) Plot the points from the table of values above and connect with a smooth curve.

(d) Copy the table of values above and complete for the following relationship.

$$y = 2^{-x}$$

(e) Plot the points from this table on the same set of axes.

(f) Copy the table of values below and complete for the following relationship.

$$y = 3 \times 2^x$$

x	-4	-3	-2	-1	0	1	2	3
y								

(g) Plot the points from this table on the same set of axes.

(h) Copy and complete the table of values below for the following relationship.

$$y = 2^{(x-3)}$$

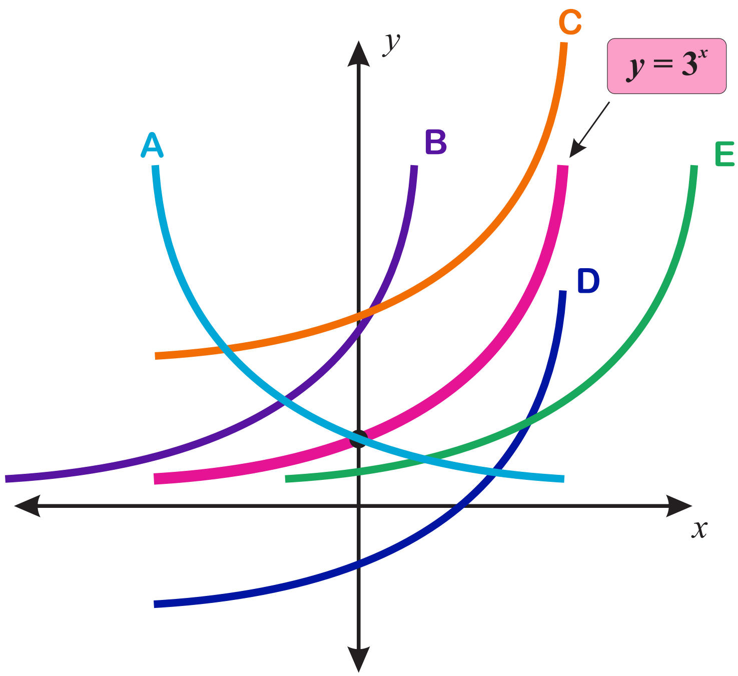
x	0	1	2	3	4
y					

(i) Plot the points from this table on the same set of axes.
(j) On the same axes plot the following relationships.

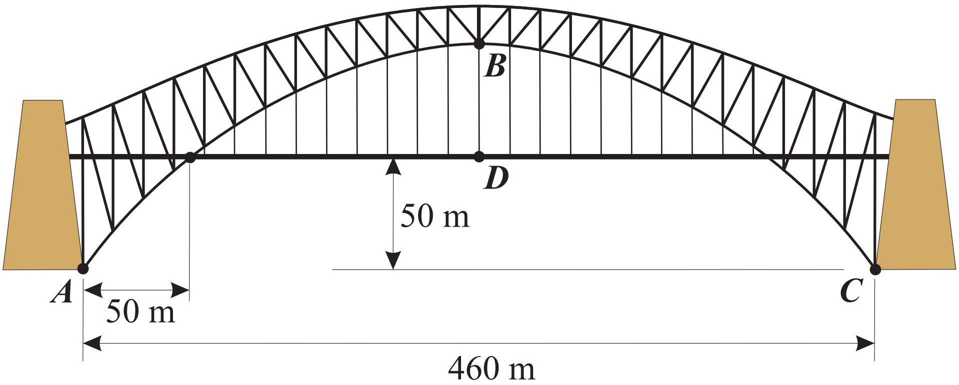
$y = 2^x + 4$ $\quad$ $y = 2^x - 4$

2. The red line on the graph below is the graph of $y = 3^x$.
Match the other lines with the following relationships.

$y = 3^{(x+2)}$ $\quad$ $y = 3^{(x-2)}$ $\quad$ $y = 3^x + 2$ $\quad$ $y = 3^{-x}$ $\quad$ $y = 3^x - 2$



PROBLEM SOLVING



A drawing of the Sydney Harbour Bridge is shown above.
The dimensions given are approximate.

1. Use the dimensions shown to write two equations that could be used to find the equation of the parabola ***ABC***. (Use point ***A*** as the origin)
2. Solve these simultaneous equations to find the equation of the parabola ***ABC***.
3. Find the length of cable ***BD***. Give answer to the nearest metre.

PUZZLE

The letters ***P***, ***A***, ***R***, ***B***, ***O*** and ***L*** represent different numbers (all greater than zero). Use the information below to find the numeral value of each letter.

$$P + A + R + A + B + O + L + A = 23$$
$$R + B = P$$
$$A + B = L$$
$$L = 2R$$

CHAPTER REVIEW

1. For each of the following equations, complete the table of values, plot the points on a graph and connect the points with a smooth curve.

(a) $y = x^2 - 4x + 3$

x	-1	0	1	2	3	4	5
y							

(b) $y = -x^2 - 2x + 1$

x	-4	-3	-2	-1	0	1	2
y							

2. (a) Without plotting points, sketch a graph of $y = x^2$.
(b) On the same graph sketch the following curves:
(i) $y = 2x^2$ (ii) $y = -x^2$ (iii) $y = \frac{1}{2}x^2$ (iv) $y = -2x^2$

3. (a) List which of the following graphs would have a ***minimum*** turning point.
(b) List which of the following graphs would have a ***maximum*** turning point.
A $y = 2x^2 + 3$ **B** $y = -5x^2$ **C** $y = \frac{1}{2}x^2 - 1$ **D** $y = \frac{1}{2}x^2 - 3x + 2$

4. Sketch the following parabolas.
(a) This parabola has - x-intercepts at -3 and 3
- y-intercept at 2

(b) This parabola has - its turning point at (1, 2)
- y-intercept at 3

(c) This parabola has - its axis of symmetry at $x = -2$
- one x-intercept at -5
- y-intercept at -1

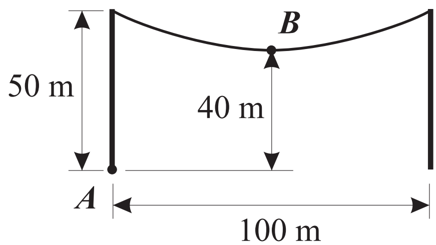
5. Find the x-intercepts, y-intercept and turning point for each of the following parabolas.
(a) $y = x^2 - 8x + 15$ (b) $y = x^2 - 16$
(c) $y = x^2 + 6x - 16$ (d) $y = x^2 - 8x$

6. Find the coordinates of the turning point for the following parabolas.
(a) $y = x^2 + 10x - 9$ (b) $y = x^2 - 5x + 8$
(c) $y = -2x^2 + 12x - 16$ (d) $y = -4x^2 - 6x + 13$

7. Sketch the following parabolas by locating the turning points and y-intercept of each.
(a) $y = 3(x - 5)^2 + 2$ (b) $y = -2(x + 3)^2 - 5$

8. A cricket ball is struck and travels 80 metres before it hits the ground. It is estimated that it reached a maximum height of 25 metres.
(a) Sketch the trajectory of the ball on coordinate axes using the origin as the point the ball was struck.
(b) Find the equation of this parabola.
(c) Find the height of the ball 10 metres from where it landed. Give answer in metres correct to one decimal place.

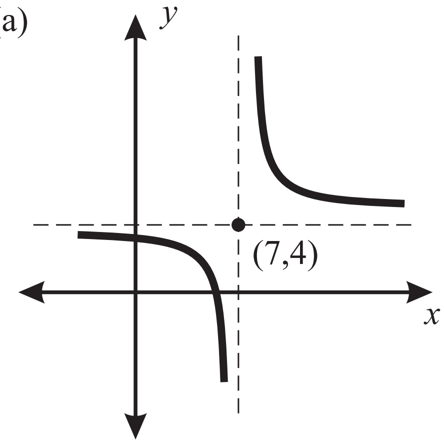
9. A cable hangs between two posts as shown in this diagram. The posts are 50 metres high and 100 metres apart. The lowest point of the cable is 40 metres above the ground.
(a) Using point A as the origin, find the equation of the parabolic shape of the cable.
(b) Copy and complete the table below that states the height of the cable every 20 metres from point A.



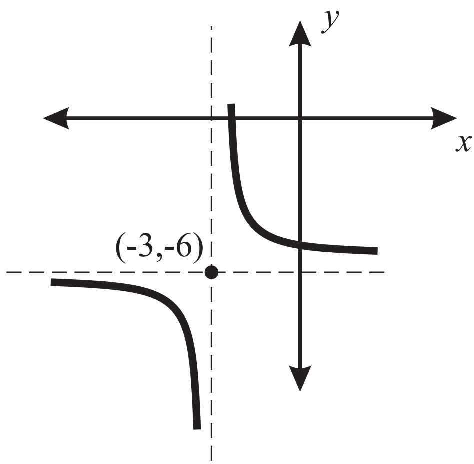
Distance from A (m)	0	20	40	60	80	100
Height above ground (m)						

10. The rectangular hyperbola with equation $\boldsymbol{y = \frac{5}{x}}$ has been moved and redrawn as shown below. Write the equation for each graph.

(a)



(b)



11. The hyperbola with equation $\boldsymbol{y = \frac{3}{x}}$ is transposed so that the intersection of the asymptotes is at the point (-6,0). Write the equation of the transposed hyperbola.

12. Write the coordinates of the point where the asymptotes intersect for the following hyperbolas.

(a) $y = \frac{3}{(x - 5)} + 4$ (b) $y = \frac{6}{x} - 2$

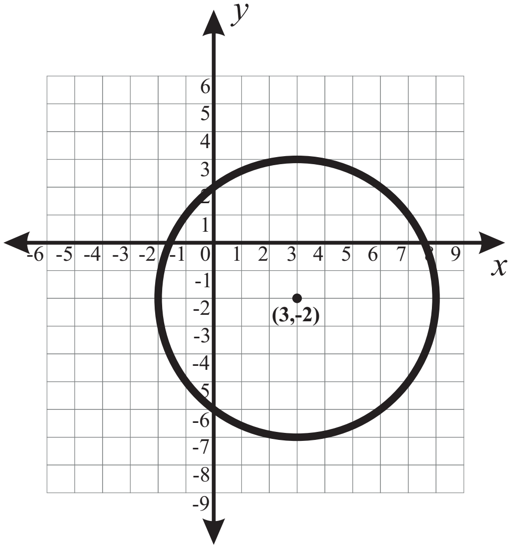
13. Write the equation of the circles with the following centres and radii.

(a) Centre (2,0), radius = 6 (b) Centre (7,-5), radius = 8

14. Write the coordinates of the centre and the radius of the circles with the following equations.

(a) $(x + 3)^2 + (y - 7)^2 = 81$ (b) $x^2 + (y + 4)^2 = 1$

15. Write the equation of the circle shown on the graph below.



16. The red line on the graph below is the graph of $y = 4^x$.
Match the other lines with the following relationships.

$y = 4^{(x-3)}$	$y = 4^{-x}$	$y = 4^{(x+3)}$	$y = 4^x - 3$	$y = 4^x + 3$